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ENERGY COMPARISONS OF WILSON-FOWLER SPLINES WITH OTHER INTERPOLATING SPLINES

Fred N. Fritsch
Lawrence Livermore National Laboratory
Livermore, CA 94550

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F. N. Fritsch

Numerical Mathematics Group
Computing & Mathematics Research Division
Lawrence Livermore National Laboratory
Livermore, CA 94550

ABSTRACT

The energy of the Wilson-Fowler spline through twenty data sets is compared with that of five other splines through the data. It is concluded that the WF-spline is sometimes better and sometimes worse than a parametric cubic spline. Areas for further research are indicated.

KEY WORDS: Wilson-Fowler splines, ν-splines, curvature continuous curves, energy.

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Wilson-Fowler Spline Energy Comparisons

1. Introduction

1.1. Background. The *Wilson-Fowler spline* (WF-spline) was introduced in the early 1960's [FoWi66] as a means for passing a smooth curve through a planar set of design points: (x_i, y_i), $i=1,\ldots,n$. The WF-spline has been used in the APT [IITR67] N/C system ever since the TABCYL (tabulated cylinder) was introduced to allow point-defined curves.

This past decade has seen the development of many CAD/CAM systems, most of which contain some sort of spline entity. The most common of these is the B-spline curve, whose component functions are (usually cubic) B-splines [deBo78] in some parameter t. Such a spline will not, in general, coincide with a WF-spline through the same data points. Thus arises the need to compare WF-splines with other splines. The original intent of this study was to answer the question: "Is the WF-spline better than an ordinary parametric cubic spline?" [A negative answer might call for a rethinking of the way parts are defined by DoE and its contractors.]

1.2. Energy as a basis for comparison. The original cubic spline arose as a mathematical model for a draftsman's spline. Here a thin beam is constrained to pass through the data points and the location of its center line at equilibrium is sought. The *elastica* is the ideal spline, which minimizes the total energy

$$(1.1) \qquad E = \int_0^S [\kappa(s)]^2 \, ds\,,$$

where s is arc length, S is the total length of the curve, and $\kappa(s)$ is the local curvature. If we regard the data as defining a function, $y = y(x)$, then

(1.2) $$ds = \sqrt{(1 + [y'(x)]^2)}\, dx\,,$$

and

(1.3) $$\kappa(x) = y''(x) \,/\, (1 + [y'(x)]^2)^{3/2}\,,$$

so the energy integral (1.1) becomes

(1.4) $$E = \int_{x1}^{xn} [y''(x)]^2 \,/\, (1 + [y'(x)]^2)^{5/2}\, dx\ .$$

If one makes the simplifying assumption that $[y'(x)]^2$ is everywhere so small relative to one that the denominator of (1.4) can be ignored, one obtains the "linearized" energy

(1.5) $$E_L = \int_{x1}^{xn} [y''(x)]^2\, dx\ .$$

It is well known [deBo78, p.63] that the function that minimizes E_L subject to the constraint of passing through the data is the *natural cubic spline* . This function $y(x)$ is a cubic polynomial on each interval $[x_i,x_{i+1}]$, the pieces are joined so that y is continuous and has continuous first and second derivatives, and it satisfies the *free-end conditions* : $y''(x_1) = y''(x_n) = 0$. [One can similarly define cubic splines that satisfy *fixed-end conditions* : $y'(x_1) = d_1$; $y'(x_n) = d_n$.]

One of the motivations for developing the WF-spline was to obtain something closer to the true elastica by introducing a local coordinate system (see Figure 1) on each segment, with the independent variable u in segment i running along the chord joining (x_i,y_i) with (x_{i+1},y_{i+1}). In such a coordinate system, it was hoped, the cubic polynomial $v(u)$, representing the deviation from the chord, would have a value of

(1.6) $$E_i = \int_{ui}^{ui+1} [v''(u)]^2\, du$$

that is closer to the true energy (1.1) of the segment than if the original independent variable x had been retained. Thus it seems natural to use the true energy as the measure of comparison between WF- and other splines.

2. Comparison Splines

2.1. Parametric piecewise cubics. Let $0 = t_1 < t_2 < \ldots < t_n$ be preselected parameter values such that a parametrized curve $\mathbf{P}(t) = (x(t), y(t))$ passes through the i-th data point at $t = t_i$:

$$(2.1) \qquad \mathbf{P}(t_i) \equiv (x(t_i), y(t_i)) = (x_i, y_i) \equiv \mathbf{P}_i, \quad i = 1,\ldots,n.$$

Such a curve is a *parametric piecewise cubic curve* if each component function is a cubic polynomial in t on each subinterval $[t_i,t_{i+1}]$. Parametric piecewise cubics have the advantage of being invariant under linear coordinate changes [deBo78, p.319]. They also allow vertical tangents, which is impossible for ordinary piecewise cubic curves. It can be shown [Frit79] that a WF-spline is a parametric piecewise cubic in the *cumulative chord length parametrization*:

$$(2.2) \qquad \begin{aligned} u_1 &= 0; \\ u_{i+1} &= u_i + L_i, \quad i=1,\ldots,n-1, \end{aligned}$$

where $L_i = \sqrt{[(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2]}$ is the length of the i-th chord. The component functions $x(u)$ and $y(u)$, however, turn out to be merely continuous. They have discontinuities in the first and second derivatives at the data points, even though the curve they describe has continuous tangent and curvature.

A *parametric cubic spline* (PC-spline) is a parametric piecewise cubic whose component functions are cubic splines. That is, $x(t)$ and $y(t)$ have continuous first and second derivatives. It is possible [Frit82] to introduce a new parametrization $t = t(u)$ for a WF-spline so that the transformed component functions $(x\hat{}(t),y\hat{}(t)) = (x(u(t)),y(u(t)))$ have continuous derivatives with respect to t. This reparametrization takes the form

$$(2.3) \qquad t = t(u) = t_i + k_i(u - u_i), \quad u \in [u_i,u_{i+1}],$$

where the u_i are as in (2.2),

$$t_1 = 0 ; \tag{2.4}$$
$$t_{i+1} = t_i + k_i (u_{i+1} - u_i) , \quad i=1,\ldots,n-1 ,$$

and the k_i are fixed constants chosen to make the derivatives continuous. Note that, due to the linear nature of (2.3), the result will be a parametric piecewise cubic in the new parameter t. It is not possible, in general, to make x^ and y^ have continuous second derivatives, so that a WF-spline is not a PC-spline.

Since we do not use arc length as the parameter for our spline, we must use the relations

$$ds = \sqrt{([x'(t)]^2 + [y'(t)]^2)}\, dt , \tag{2.5}$$

and

$$\kappa(t) = \pm \frac{x'(t)\, y''(t) - y'(t)\, x''(t)}{([x'(t)]^2 + [y'(t)]^2)^{3/2}} , \tag{2.6}$$

to compute the true energy (1.1) as

$$E = \sum_1^{n-1} \int_{t_i}^{t_{i+1}} (x' y'' - y' x'')^2 / (x'^2 + y'^2)^{5/2}\, dt . \tag{2.7}$$

2.2. ν-splines. In order to allow for the application of local tension at the data points, Nielson [Niel74] defined a *ν-spline* to be the function f(t) that interpolates given data (t_i, f_i), $i=1,\ldots,n$, and minimizes the functional

$$E_\nu[f] = \int_{t_1}^{t_n} [f''(t)]^2\, dt + \sum_1^n \nu_i [f'(t_i)]^2 , \tag{2.8}$$

where the ν_i are fixed (nonnegative) *tension parameters* . We recognize the first term as being the linearized energy (1.5) that is minimized by the cubic spline, so a ν-spline will generally have only first derivative continuity. In fact, a ν-spline is characterized by the *jump conditions*

$$f''(t_i+) - f''(t_i-) = \nu_i f'(t_i) , \tag{2.9}$$

from which we see that the ordinary cubic spline is the special ν-spline with $\nu_i = 0$, $i=1,\ldots,n$.

A *parametric ν-spline* is a parametric piecewise cubic, each of whose component functions is a ν-spline (with the same ν-values for each component). A parametric ν-spline has continuous tangent and curvature [Niel74]. A PC-spline is a special parametric ν-spline with $\nu_i = 0$, $i=1,\ldots,n$. It can be shown [Frit85] that a WF-spline is a parametric ν-spline, provided one takes the jump conditions (2.9) as the defining relation and drops the restriction $\nu_i \geq 0$. Thus, all of the splines of interest to us are ν-splines, for a suitable choice of parametrization and ν_i.

One may define a *uniformly-shaped ν-spline* to be one with $\nu_i=\nu$, $i=1,\ldots,n$. Thus, a PC-spline is a special uniformly-shaped ν-spline with $\nu=0$. It is of interest to compare the energy of the WF-spline and the PC-spline with that of the *optimal uniformly-shaped ν-spline* (OUSN-spline). This is the uniformly-shaped ν-spline that has the minimum value of the true energy E as computed from (2.7). [We have chosen this ν-spline simply because it is computable via a univariate optimization algorithm and gives some indication of the improvement possible over a PC-spline. There is no reason to believe that choosing all ν_i equal is a good idea, but we have not attempted minimizing over all possible ν_i-values.]

3. The Tests

3.1. The test data. The twenty data sets employed in these tests were as follows:

- `SIN1` : This is the sample data in [FoWi66]; namely, $y_i = 2\cdot\sin(x_i)$, for 28 uniformly-spaced x-values in $[0,3\pi/2]$
- `SIN2` : This is every third point from `SIN1`.

QCIR1 : This is a set of uniformly-spaced points on the quarter circle with radius 1.5 and angles in [0,90°]. (The points were read in polar coordinates, at 5° increments, and converted to cartesian coordinates internally.)

QCIR2 : This is every other point from QCIR1.

WRM[1] : This is a constructed data set used by W. R. Melvin when he was testing his algorithm [Melv82] for computing a WF-spline.

SF1,SF2[1]: These are two constructed data sets used by S. K. Fletcher as part of her spline testing procedure [Flet83].

FNFk, k=1,...,5[1]: This is a series of five test sets constructed by the author. (They were originally invented to visually demonstrate the derivative discontinuities of the WF-spline component functions, since most "reasonable" data sets yield derivative jumps smaller than 10^{-3}.)

RPN[1]: This data set was constructed by the author so that $(u_i+7.99, x_i)$, with u_i given by (2.2), is the RPN14 data set of [FrCa80].

BMK1[1]: This is a set of data that has been use by LLNL as a benchmark for vendor-supplied splines. The points are taken from a pair of tangent ellipses and contain one inflection. They are given in polar coordinates at 2° increments.

BMK2[1]: This is a subset of BMK1, retaining its original character.

JJi,JMj: These are five actual design contours All but the last are very similar to BMK1.

3.2. Parametrizations. We have mentioned previously two "natural" parametrizations for splines. The first is the cumulative chord length parametrization (2.2), which will be called the *natural parametrization*. The second is the parametrization (2.4) required to give the WF-spline component functons continuous derivatives, called the *WF parametrization*. The five splines compared with the WF-spline in this study were the PC-

[1] These data are listed in the appendix.

spline and OUSN-spline using each of these parametrizations and the ordinary (non - parametric) cubic spline.

3.3. Boundary conditions. In order to completely determine a parametric ν-spline, it is necessary to specify some sort of boundary conditions (BC). The BC employed in these tests were specified tangents $(x'(t_1), y'(t_1))$ and $(x'(t_n), y'(t_n))$. Since a WF-spline is determined simply by end-slopes, there is some arbitrariness in the choice of the magnitudes of the boundary tangent vectors. We choose to use the formulas

$$x'(u_1) = L_1 / [(x_2-x_1) + S_1 \cdot (y_2-y_1)] \;,$$

(3.1)

$$y'(u_1) = S_1 \cdot x'(u_1) \;,$$

and their analog at u_n, which arise naturally when one represents a WF-spline as a parametric piecewise cubic [Frit79]. (In (3.1), L_1 and S_1 are the length and slope of the first chord.) These BC are scaled appropriately for the transformation to the WF parametrization.

For the sine data, two different sets of BC were used. One is the "correct" boundary slopes: $S_1=2$, $S_n=0$. The other is the "default" BC, namely that each boundary slope match that of the circle through the three end points. (Note that this is close to, but not the same as, the default BC of [FoWi66, pp.21-22].) For the other data sets, a single BC was chosen, making a total of 22 tests in all.

4. Test Results

The results of these tests, as run on a CRAY-1 using single precision arithmetic, are given in Tables 1 and 2. The default BC are indicated by "`Def.`". The notation "`N0S0`" means the curve has a zero normal (vertical slope) at (x_1,y_1) and a zero slope at (x_n,y_n). Similarly, "`S2S0`" indicates an intital slope of two and a final slope of zero. All reported energies were computed using a 50-panel Simpson's rule to approximate each of the $n-1$ integrals in (2.7). (Numerical tests indicate that the result is probably accurate to at least five decimal places.) The "optimal ν", ν_{opt}, was computed by subroutine LCLMIN [Haus72], with a requested final interval length of 10^{-3}. (Tests showed that the results are relatively insensitive to this convergence criterion.) Further details on the computations may be found in [FrSp85].

Table 1 contains details on the WF-spline through each of these data sets, with the indicated BC. Where possible, the ordinary cubic spline was also obtained, and its energy E_0 computed via (1.4). The other quantities in the table are the total arc length and energy of the curve; k_{ratio}, the ratio of the largest and smallest k_i in (2.4); ν_{min} and ν_{max}, the largest and smallest ν_i-values, when represented as a ν-spline in the WF parametrization.

Table 2 compares the WF-spline with the four other parametric ν-splines discussed in Section 3.2. The quantity $\Delta E = E - E_{WF}$ will be positive if, and only if, the WF-spline has the smaller energy. (Due to the probable accuracy of the integrals, ΔE is given only to one significant figure if it is smaller than 10^{-6} in magnitude.) The value of ν_{opt} is also given for each of the OUSN-splines. In case $k_{ratio}=1$, no reparametrization was necessary, and the last three columns have been left blank.

Wilson-Fowler Spline Energy Comparisons

Table 1. Results for WF-splines

Data ID	n	BC	E_o	E_{WF}	arc length	k_{ratio}	v_{min}	v_{max}
SIN1	28	S2S0	3.7339	3.7337	7.9506	1.0075	-5.56	0.
SIN1	28	Def.	3.6346	3.6341	7.9055	1.0075	-5.56	0.
SIN2	10	S2S0	3.7360	3.7497	7.9116	1.0462	-14.44	0.
SIN2	10	Def.	3.0508	3.0156	7.9023	1.0494	-14.37	0.
QCIR1	19	N0S0	3.2624[2]	1.0472	2.3562	1.0000	-0.00	-0.00
QCIR2	10	N0S0	4.2967[2]	1.0472	2.3562	1.0000	-0.00	-0.00
WRM	4	Def.	.--- [3]	4.9150	2.4966	1.1945	-3.56	0.
SF1	5	Def.	2.0696[4]	0.7826	7.9263	1.1070	-3.16	-0.51
SF2	7	Def.	17.769 [4]	1.0648	6.6100	1.0429	-3.10	+0.10
FNF1	6	S1S1	1.5657[4]	1.4088	13.738	1.0838	-5.63	-2.49
FNF2	6	S1S1	4.7422[4]	1.6652	14.524	1.2505	-6.15	-2.70
FNF3	6	S1S1	45.099 [4]	2.6315	16.034	1.9302	-9.56	+0.61
FNF4	6	S1S1	245.19 [4]	3.2239	16.779	2.3239	-10.08	+1.80
FNF5	6	S1S1	.--- [3]	3.3066	16.881	2.3777	-10.13	+1.94
RPN	9	Def.	14.188 [5]	5.2661	12.059	1.1551	-38.65	+0.05
BMK1	46	Def.	1.1056[6]	0.5654	6.2650	1.0006	-0.26	-0.01
BMK2	11	Def.	3.6613[6]	0.5641	6.2649	1.0097	-1.03	-0.04
JJ2	46	Def.	0.6278[6]	0.5427	4.6008	1.0001	-0.07	-0.01
JJ3	46	Def.	1.2322[6]	0.5451	4.6382	1.0002	-0.08	-0.01
JM1	46	Def.	1.1764[6]	0.4049	6.4864	1.0003	-0.13	-0.00
JM2	46	Def.	1.0352[6]	0.5449	6.7003	1.0008	-0.42	+0.00
JM3	85	Def.	0.2664	0.2664	16.298	1.0003	-0.27	0.

[2] This is the "natural" spline. (Simulations of vertical slope yielded astronomical energy values.)

[3] Could not compute spline: two data points have same x-coordinate.

[4] The "natural" spline has smaller energy, also greater than the WF-spline.

[5] This is the "natural" spline. (Default BC gave energy ca. 1.7×10^7.)

[6] This is the "natural" spline. (Default BC gave energies in excess of 3000, due to a nearly vertical end slope.)

Table 2. Energy Comparisons

Data ID	BC	E_{WF}	Natural parametrization: ΔE for $\nu=0$	Natural parametrization: ΔE for ν_{opt}	Natural parametrization: ν_{opt}	WF parametrization: ΔE for $\nu=0$	WF parametrization: ΔE for ν_{opt}	WF parametrization: ν_{opt}
SIN1	S2S0	3.7337	-1.1E-4	-1.8E-4	-1.15	-1.2E-4	-1.9E-4	-1.18
SIN1	Def.	3.6341	-6.1E-5	-1.5E-4	-1.28	-8.5E-5	-1.7E-4	-1.26
SIN2	S2S0	3.7497	+2.5E-2	+1.6E-2	-4.37	+2.0E-2	+1.0E-2	-4.41
SIN2	Def.	3.0156	+2.9E-2	+1.5E-2	-5.15	+2.2E-2	+8.2E-3	-5.04
QCIR1	N0S0	1.0472	-3. E-7	-5. E-7	-0.05			
QCIR2	N0S0	1.0472	-4.9E-6	-7.8E-6	-0.10			
WRM	Def.	4.9150	-3.9E-2	-4.1E-2	-0.73	+1.5E-2	+1.4E-2	+0.64
SF1	Def.	0.7826	+7.2E-4	-4.9E-3	-2.06	+1.6E-3	-4.2E-4	-1.21
SF2	Def.	1.0648	-4.1E-4	-1.9E-3	-1.41	-6.7E-4	-2.8E-3	-1.69
FNF1	S1S1	1.4088	+4.5E-3	-6.5E-3	-2.93	-4.9E-3	-9.2E-3	-1.75
FNF2	S1S1	1.6652	-2.1E-2	-4.9E-2	-4.03	-3.6E-2	-3.6E-2	-0.29
FNF3	S1S1	2.6315	-5.6E-1	-6.5E-1	-6.10	-2.8E-1	-4.2E-1	+9.12
FNF4	S1S1	3.2239	-1.0	-1.1	-6.62	-4.1E-1	-8.0E-1	+17.0
FNF5	S1S1	3.3066	-1.1	-1.2	-6.67	-4.3E-1	-8.6E-1	+18.2
RPN	Def.	5.2661	-8.9E-4	-3.9E-3	-3.73	+1.8E-3	+1.5E-3	-1.30
BMK1	Def.	0.5654	-3. E-8	-5. E-8	-0.05	---------------[7]		
BMK2	Def.	0.5641	+6.0E-6	-9.9E-6	-0.31	-2. E-7	-1.6E-5	-0.30
JJ2	Def.	0.5427	-5. E-9	-8. E-9	-0.02	---------------[7]		
JJ3	Def.	0.5451	-6. E-9	-9. E-9	-0.02	---------------[7]		
JM1	Def.	0.4049	-6. E-9	-9. E-9	-0.03	---------------[7]		
JM2	Def.	0.5449	+3. E-8	+2.E-11	-0.07	-4. E-8	-7. E-8	-0.07
JM3	Def.	0.2664	-4. E-9	-7. E-9	-0.05	-5. E-9	-7. E-9	-0.05

[7] Results identical to those for natural parametrization to all digits given.

5. Observations and Conclusions

5.1. The majority of the ν-values given in the tables are negative. This means that the curve is "looser" than the corresponding parametric cubic spline. It also means that the curve may not minimize functional (2.8).

5.2. The WF-spline usually has a smaller energy than the ordinary cubic spline, as predicted [Fowl61]. The only exception is the SIN2 data with specified boundary slopes (see Table 1).

5.3. For "realistic" data sets, the WF-spline appears to be as good as, or marginally better than the PC-spline. Furthermore, the WF-spline component functions have extremely small derivative discontinuities for such data.

5.4. For one data set (SIN2) the WF-spline is better than both the PC-spline and the OUSN-spline[8] in either parametrization. For two sets (WRM and RPN) it is better than both in the WF parametrization, but worse than both in the natural parametrization. For another set (JM2) it is (marginally) better than both in the natural parametrization, but worse than both in the WF parametrization. For one (SF1) it is better than PC but worse than OUSN in both parametrizations. For two (FNF1, BMK2) it is better than the PC-spline only in the natural parametrization. In all these cases except WRM and RPN, the five parametric splines are so similar as to be indistinguishable on the scale of a plot. Comparing Figures 2 and 3, for example, there is no obvious reason why the WF-spline should be better for one than the other.

5.5. On the other hand, there are some data sets (FNF3-FNF5) for which the WF-spline is much worse than the PC spline, and the natural parametrization is significantly

[8] This is possible, since the WF-spline is definitely <u>not</u> uniformly shaped.

better than the WF parametrization for the same choice of ν. This is clearly illustrated by Figures 4-9, in which the six splines through the FNF4 data set are seen to be clearly different.

5.6. Considering the amount of energy reduction achievable over the PC-spline, it is probably not worth the effort to compute ν_{opt}.

6. Open Questions

It is clear that much more work is needed to supply the "why?" for the conclusions of the previous section. Some of the open questions include the following.

6.1. Why is the WF-spline so good on the `SIN2` data and so poor on the `FNF` data? For what types of data sets can we expect the WF-spline to be good?

6.2. Since we observed such a dramatic change in energy and curve shape with parametrization in the `FNF` data sets, it might be worth investigating "optimal parametrizations" for ν-splines.

6.3. We have not investigated at all the effect of the magnitudes of the boundary tangent vectors on the energy of the curve.

6.4. The `FNF` data are really five representatives of a parametrized family of data sets (see Appendix), the parameter being μ, the magnitude of the abscissae of the third and fourth points. It is not evident from the data presented in Table 1, but there are really only two distinct ν-values, due to the symmetry of the data. Their values, together with the values of μ and NIT, the number of iterations of Melvin's algorithm [Melv82] required for computing the WF parameters, for the `FNFk` data sets are given in the following table:

Wilson-Fowler Spline Energy Comparisons

k	μ	ν_2	ν_3	NIT
1	1.0	-2.49	-5.63	3
2	0.5	-6.15	-2.70	4
3	0.1	-9.60	+0.61	6
4	0.01	-10.08	+1.80	15
5	0.	-10.13	+1.94	9

Among the questions that suggest themselves are:

a. If the value of $\mu \in (0.5,1.0)$ at which $\nu_2=\nu_3$ were chosen, how would the energy of the WF-spline (which would be uniformly shaped) compare with that of the OUSN-spline?

b. What happens when $\nu_3=0$?

c. Why isn't NIT monotonic?

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[9] In my early work, I referred to the WF-spline as the FW-spline, due to the order of the authors in [FoWi66]. WF is evidently the accepted order.

[FrCa80] F. N. Fritsch and R. E. Carlson, "Monotone piecewise cubic interpolation", *SIAM J. Numer. Anal.* **17**, 2 (April 1980), 238-246.

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Appendix. Listing of the Test Data.

WRM:

These four points are: (0, 2), (0.5, 1.5), (1, 1), (1, 0).

SF1:

These five points are: (5, 0), (4, 1), (3, 4), (1, 5), (0, 5).

SF2:

These seven points are: (-2.5, 1), (-2, 2), (-1, 2.5), (0, 2.75), (1, 2.5), (2, 2), (2.5, 1).

FNFk, k=1,...,5:

These six points are: (-5, 2), (-3, 3), ($-\mu_k$, 2), (μ_k, -2), (3, -3), (5, -2), where $\mu_1=1$, $\mu_2=0.5$, $\mu_3=0.1$, $\mu_4=0.01$, $\mu_5=0$.

RPN:

These nine points are:

i	x_i	y_i
1	0.	0.
2	0.000027	0.1
3	0.043722	0.189935
4	0.169183	0.684269
5	0.469428	1.084085
6	0.943740	1.728312
7	0.998636	3.727558
8	0.999919	6.727558
9	0.999994	11.727558

Wilson-Fowler Spline Energy Comparisons

BMK1 (BMK2):

These data are listed below, in polar coordinates. (θ is in degrees.) Point numbers in parentheses indicate which points of BMK1 constitute BMK2.

i	θ_i	r_i	i	θ_i	r_i
1(1)	0.	4.582575	24	46.	3.774696
2	2.	4.581565	25	48.	3.717856
3	4.	4.578537	26(6)	50.	3.662133
4	6.	4.573497	27	52.	3.607701
5	8.	4.566457	28	54.	3.554702
6(2)	10.	4.557429	29	56.	3.503250
7	12.	4.546433	30	58.	3.453436
8	14.	4.533491	31(7)	60.	3.405328
9	16.	4.516018	32	62.	3.358977
10	18.	4.491610	33	64.	3.314416
11(3)	20.	4.460876	34	66.	3.271665
12	22.	4.424473	35	68.	3.230735
13	24.	4.383080	36(8)	70.	3.191622
14	26.	4.337383	37	72.	3.154470
15	28.	4.288056	38	74.	3.123414
16(4)	30.	4.235752	39	76.	3.099905
17	32.	4.181087	40(9)	78.	3.083516
18	34.	4.124637	41	80.	3.073963
19	36.	4.066934	42	82.	3.071082
20	38.	4.008456	43(10)	84.	3.074826
21(5)	40.	3.949636	44	86.	3.085257
22	42.	3.890854	45	88.	3.102553
23	44.	3.832444	46(11)	90.	3.127017

Figure Captions

Figure 1. Wilson-Fowler spline local coordinate system.

Figure 2. WF-spline through `SIN2` data (default BC).

Figure 3. WF-spline through `BMK2` data (default BC).

Figure 4. Ordinary cubic spline through `FNF4` data (default BC). (Note the drastically different vertical scale than for Fig. 5-9.)

Figure 5. WF-spline through `FNF4` data (default BC).

Figure 6. PC-spline through `FNF4` data (default BC; natural parametrization).

Figure 7. OUSN-spline through `FNF4` data (default BC; natural parametrization).

Figure 8. PC-spline through `FNF4` data (default BC; WF parametrization).

Figure 9. OUSN-spline through `FNF4` data (default BC; WF parametrization).

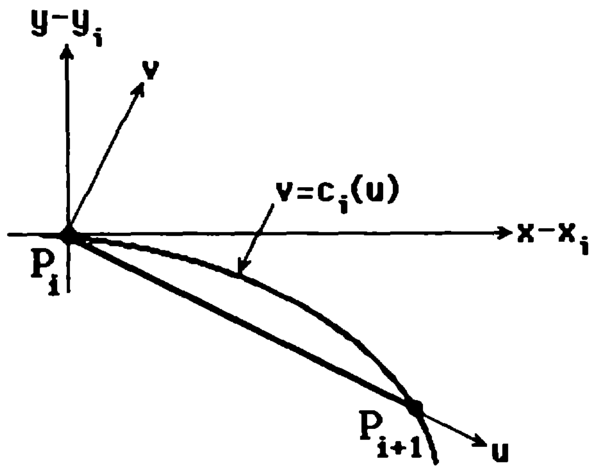

Figure 1. Wilson-Fowler spline local coordinate system.

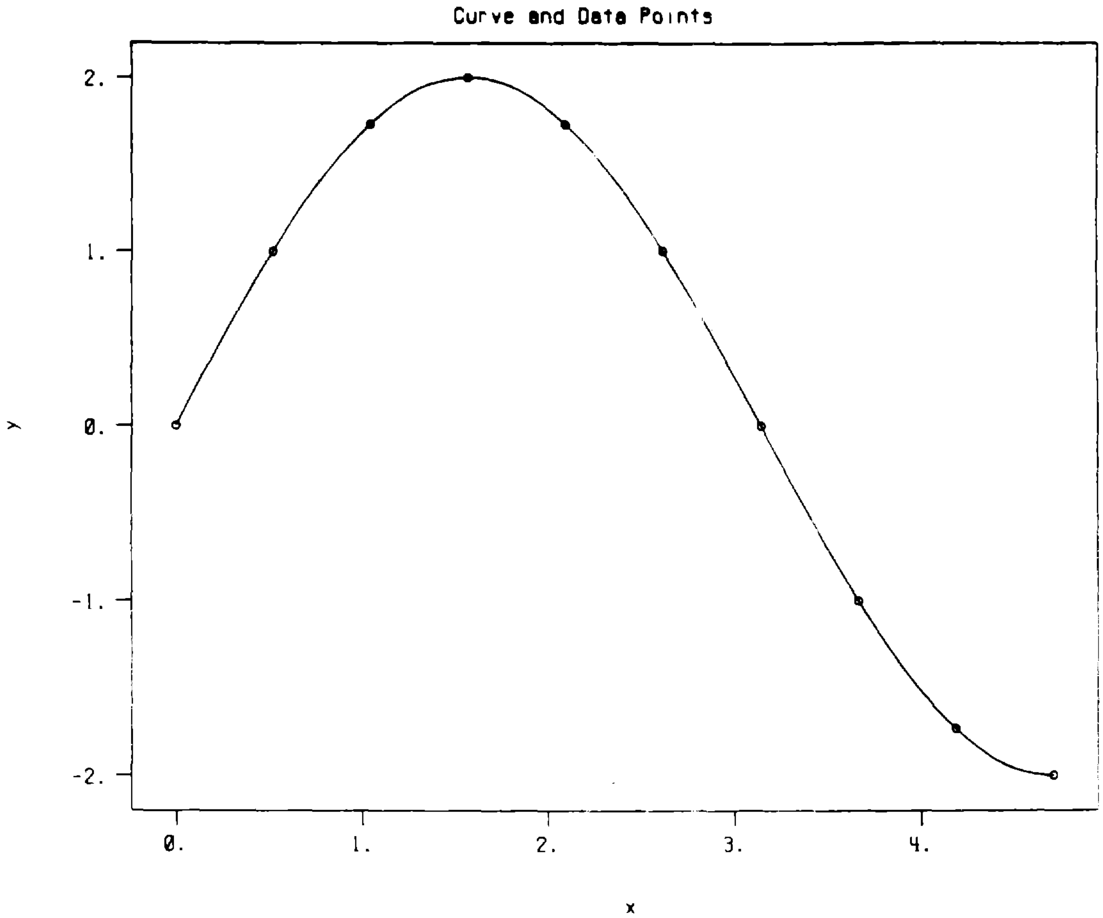

Figure 2 WF-spline through SIN2 data (default BC).

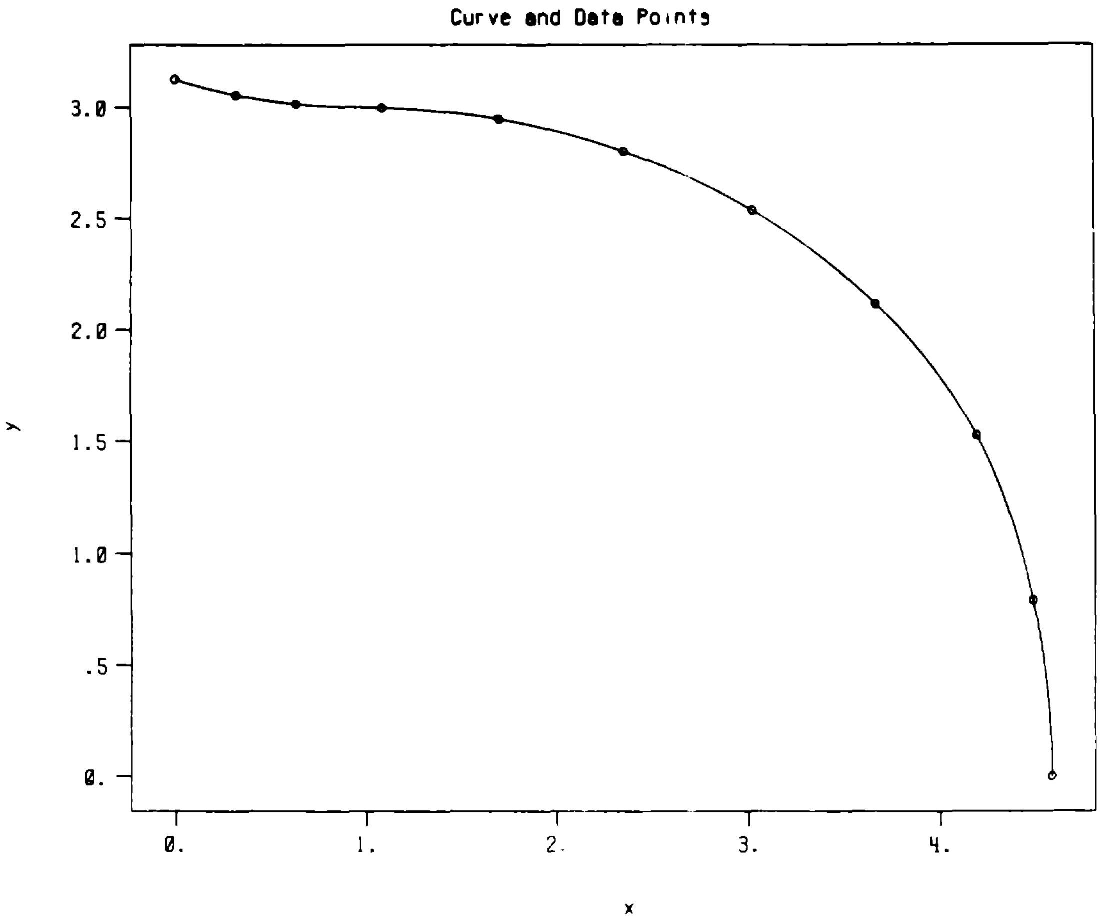

Figure 3. WF-spline through BMK2 data (default BC).

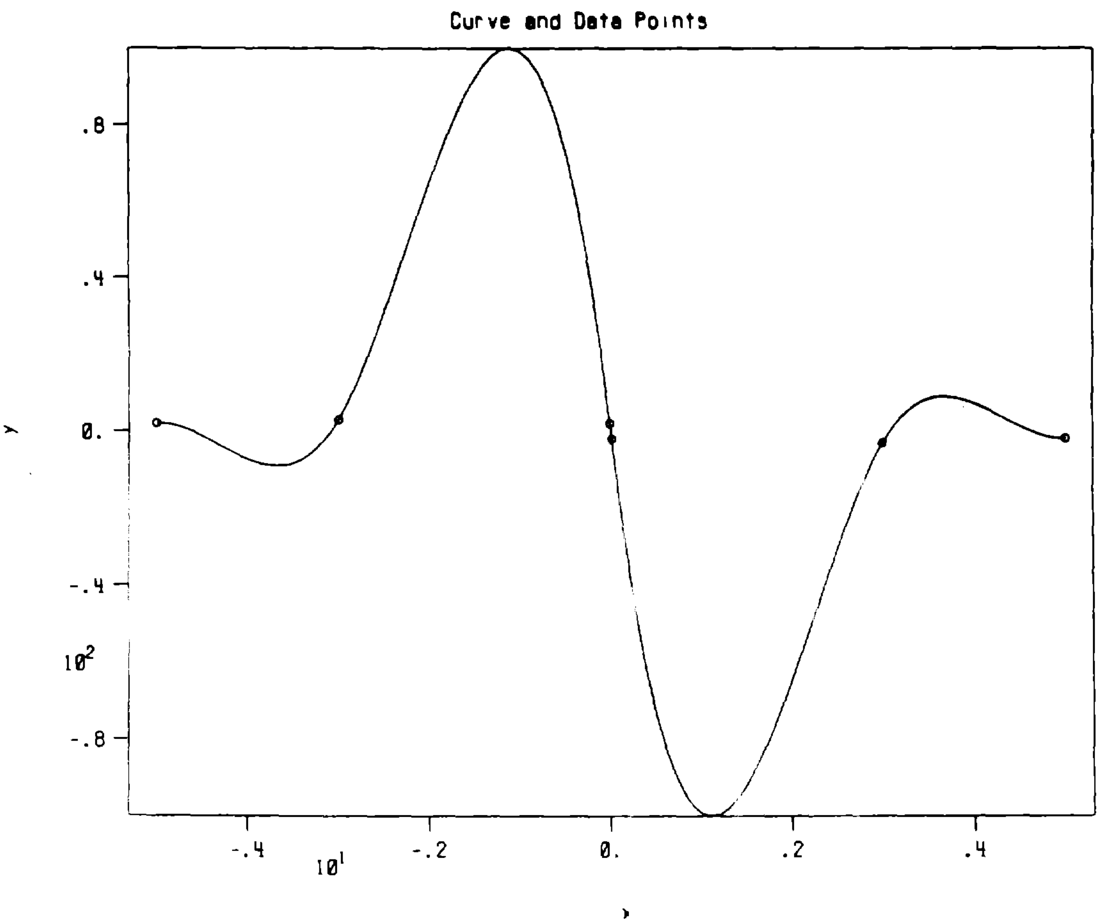

Figure 4. Ordinary cubic spline through FNF4 data (default BC).

(Note the drastically different vertical scale than for Fig. 5-9.)

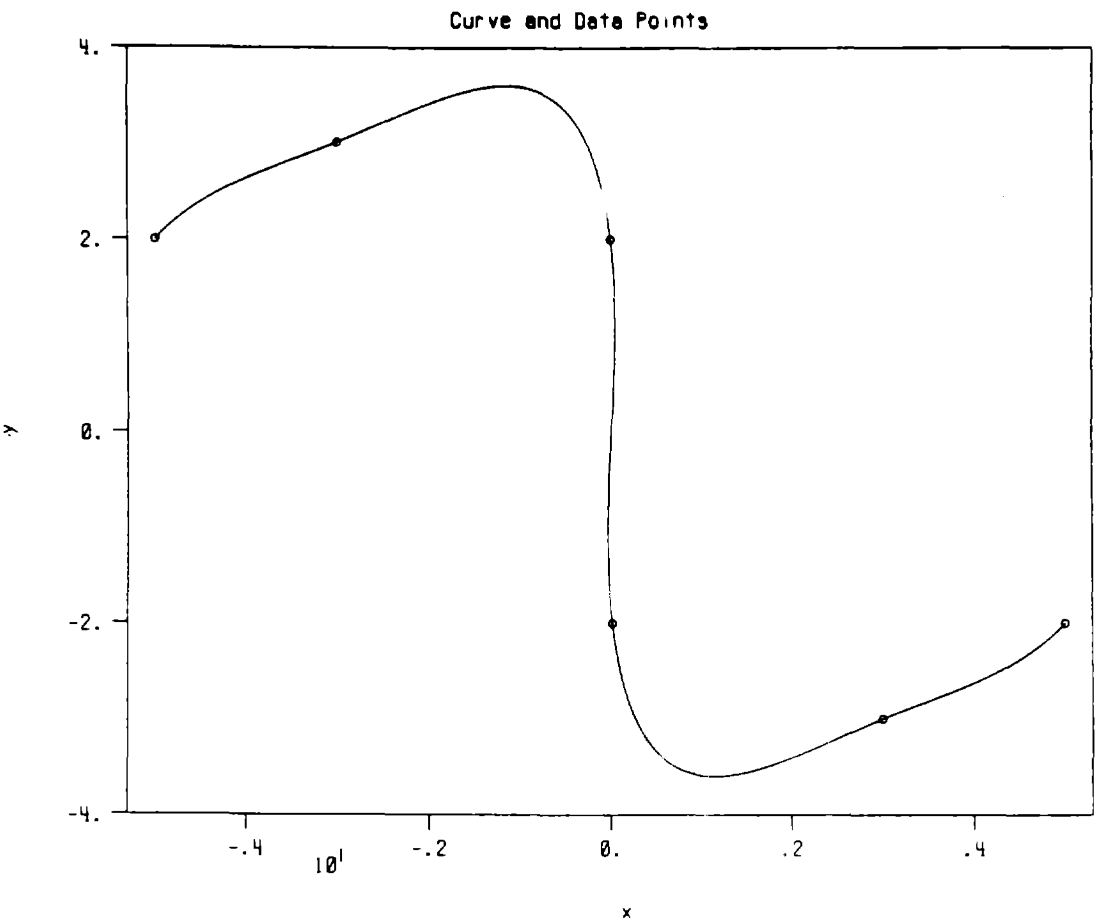

Figure 5. WF-spline through FNF4 data (default BC).

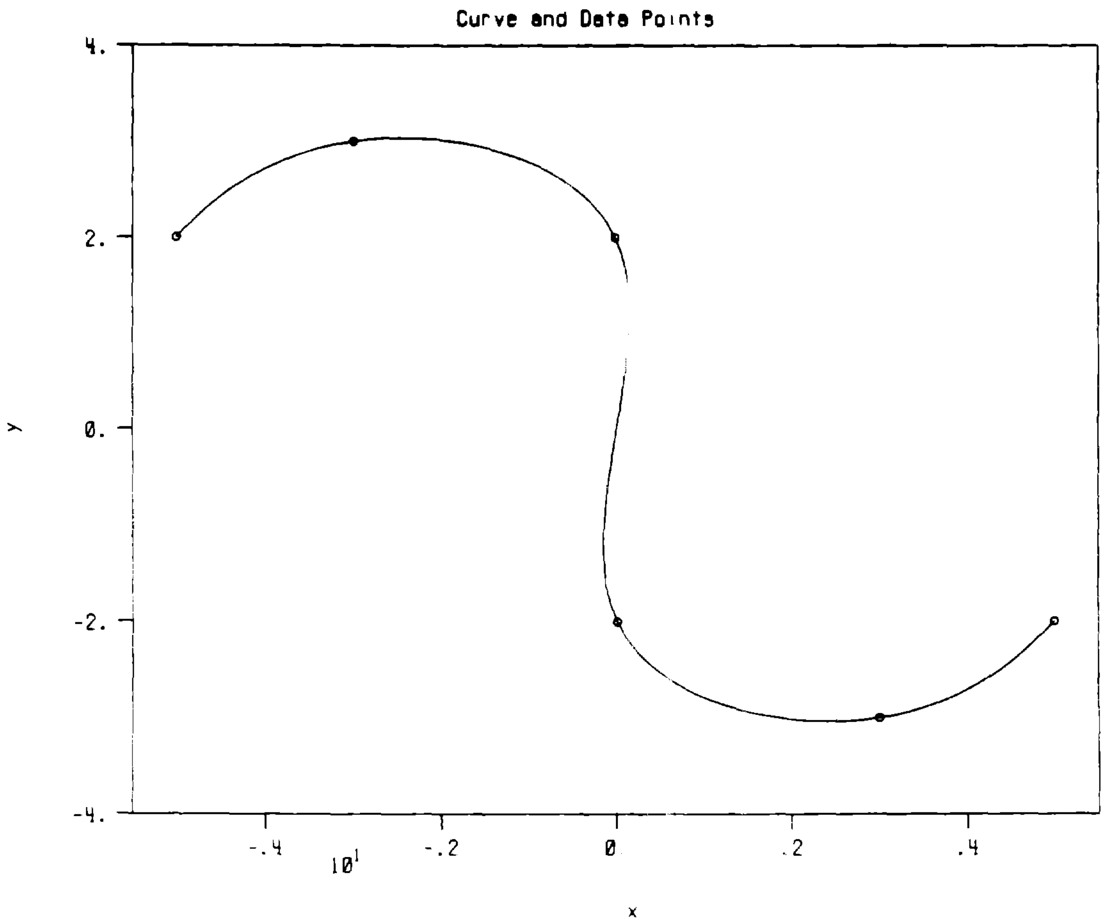

Figure 6. PC-spline through FNF4 data

(default BC; natural parametrization).

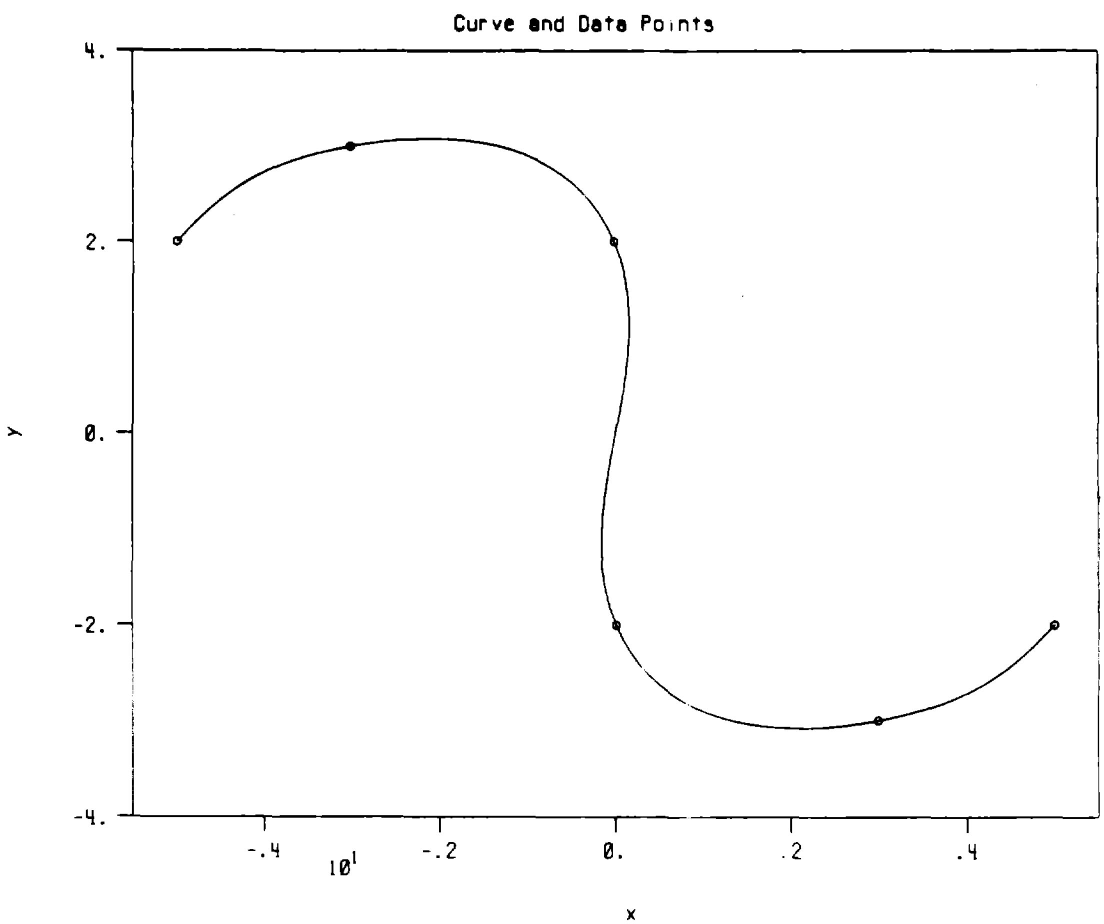

Figure 7. OUSN-spline through FNF4 data

(default BC; natural parametrization).

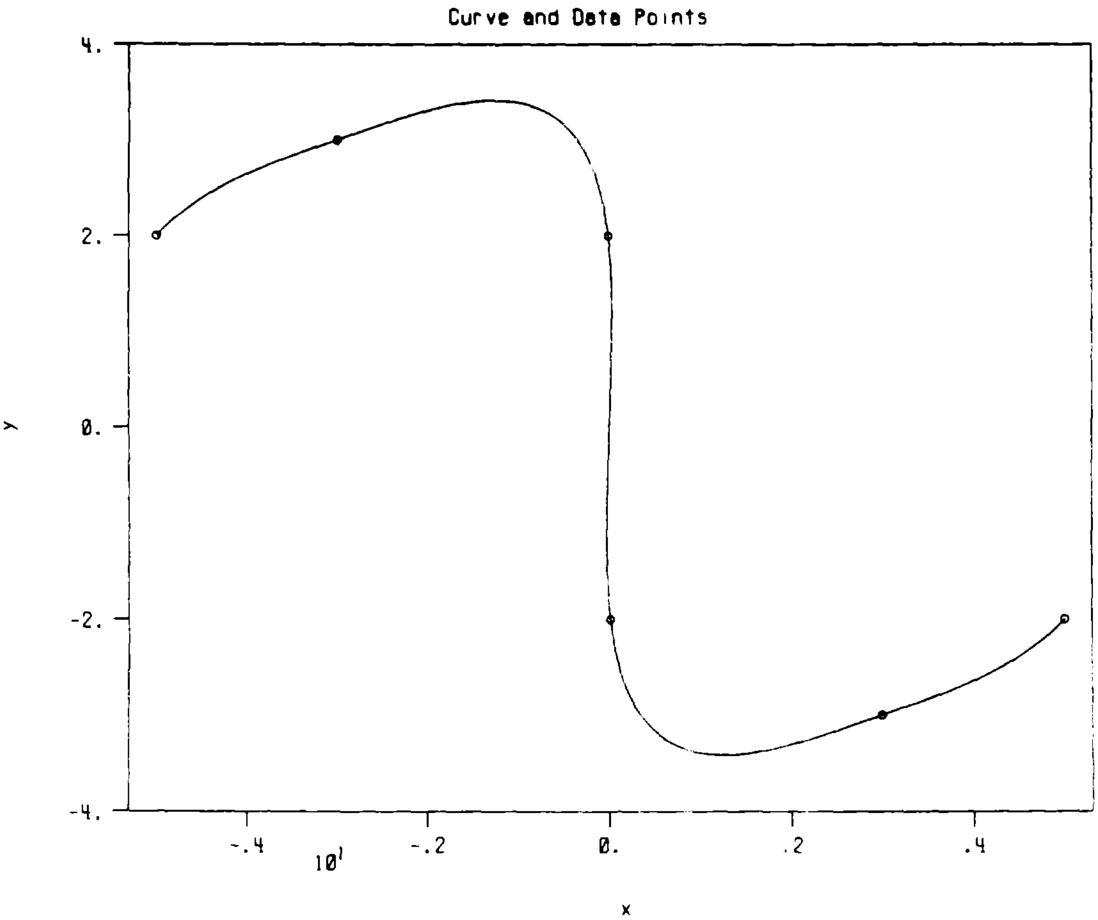

Figure 8. PC-spline through FNF4 data (default BC; WF parametrization).

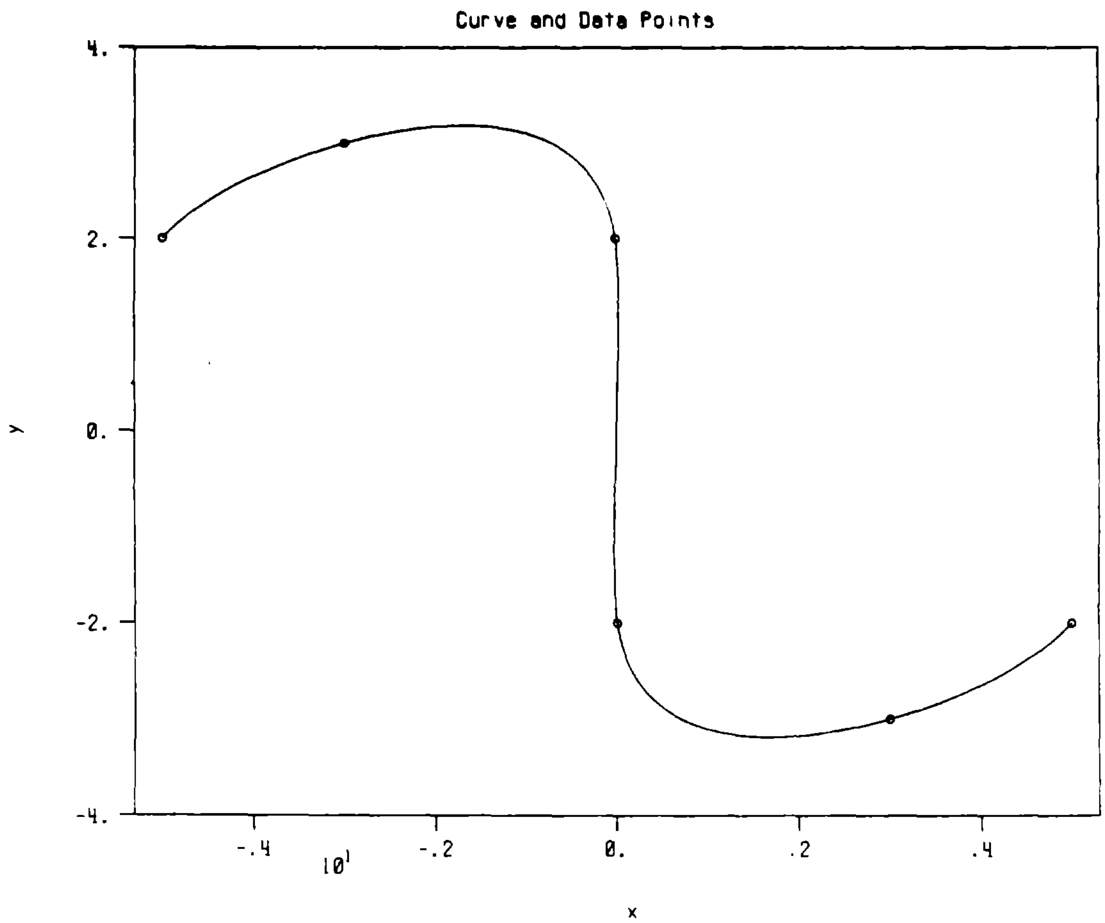

Figure 9. OUSN-spline through FNF4 data (default BC; WF parametrization).

Footnotes

* This work was performed under the auspices of the U. S. Department of Energy by Lawrence Livermore National Laboratory under Contract W-7405-Eng-48. It was supported in part by the Applied Mathematics Research Program, Office of Energy Research.

1 These data are listed in the appendix.

2 This is the "natural" spline. (Simulations of vertical slope yielded astronomical energy values.)

3 Could not compute spline: two data points have same x-coordinate.

4 The "natural" spline has smaller energy, also greater than the WF-spline.

5 This is the "natural" spline. (Default BC gave energy ca. 1.7×10^7.)

6 This is the "natural" spline. (Default BC gave energies in excess of 3000, due to a nearly vertical end slope.)

7 Results identical to those for natural parametrization to all digits given.

8 This is possible, since the WF-spline is definitely <u>not</u> uniformly shaped.

9 In my early work, I referred to the WF-spline as the <u>FW</u>-spline, due to the order of the authors in [FoWi66]. WF is evidently the accepted order.